Self-Duality for the Two-Component Asymmetric Simple Exclusion Process

V. Belitsky[1]   G.M. Schütz[2,3]

April 21, 2015

[1]Instituto de Matemática e Estatística, Universidade de São Paulo, Rua do Matão, 1010, CEP 05508-090, São Paulo - SP, Brazil
Emails: belitsky@ime.usp.br

[2]Institute of Complex Systems II, Forschungszentrum Jülich, 52425 Jülich, Germany
Email: g.schuetz@fz-juelich.de

[3]Interdisziplinäres Zentrum für Komplexe Systeme, Universität Bonn, Brühler Str. 7, 53119 Bonn, Germany
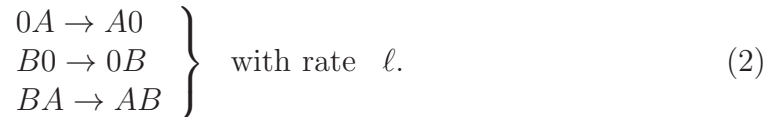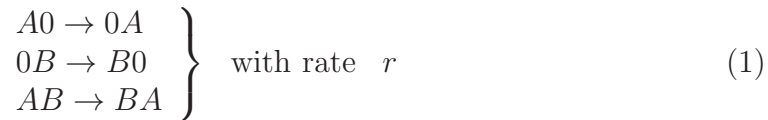URL: http://www.izks.uni-bonn.de

Abstract

We study a two-component asymmetric simple exclusion process (ASEP) that is equivalent to the ASEP with second-class particles. We prove self-duality with respect to a family of duality functions which are shown to arise from the reversible measures of the process and the symmetry of the generator under the quantum algebra $U_q[\mathfrak{gl}_3]$. We construct all invariant measures in explicit form and discuss some of their properties. We also prove a sum rule for the duality functions.

Keywords: Asymmetric simple exclusion process; Second Class Particles; Duality; Quantum algebras

AMS 2000 subject classifications: 82C20. Secondary: 60K35, 82C23

1 Introduction

We consider an asymmetric simple exclusion process with two species of particles on the one-dimensional finite lattice $\Lambda = \{-L+1, \ldots, L\}$. Its Markovian dynamics can be described informally as follows. Each site i can be either empty (denoted by 0) or occupied by at most one particle of type A or of type B. Thus we have local occupation numbers $\eta(k) \in \{A, 0, B\}$. We define the bonds $(k, k+1)$ of Λ where $-L+1 \leq k \leq L-1$. Each bond carries a clock i which rings independently of all other clocks after an exponentially distributed random time with parameter τ_k where $\tau_k = r$ if $(\eta(k), \eta(k+1)) \in \{(A, 0), (0, B), (A, B)\}$ and $\tau_k = \ell$ if $(\eta(k), \eta(k+1)) \in \{(0, A), (B, 0), (B, A)\}$. When the clock rings the particle occupation variables are interchanged and the clock acquires the corresponding new parameter. Symbolically this process can be presented by the nearest neighbour particle jumps

$$
\left.
\begin{aligned}
A0 &\to 0A \\
0B &\to B0 \\
AB &\to BA
\end{aligned}
\right\} \quad \text{with rate} \quad r \tag{1}
$$

$$
\left.
\begin{aligned}
0A &\to A0 \\
B0 &\to 0B \\
BA &\to AB
\end{aligned}
\right\} \quad \text{with rate} \quad \ell. \tag{2}
$$

We have reflecting boundary conditions, which means that no jumps from the left boundary site $-L+1$ to the left and no jumps from the right boundary site L to the right are allowed. We shall assume partially asymmetric hopping, i.e., $0 < r, \ell < \infty$. By interchanging the B-particles and vacancies this process turns into the ASEP with second-class particles [1]. We choose an even number of lattice sites exclusively for the sake of convenience of notation.

The objective of this work is to construct for the finite lattice in explicit form all reversible measures and to prove self-duality with respect to a family of duality functions that allows for the computation of expectations of the many-particle system in terms of transition probabilities of the same process with only a small number of particles. It will transpire that this property, analogous to the well-known self-duality of the simple symmetric exclusion process [2], arises from the fact proved in [3] that the generator of this process commutes with a set of matrices which form a representation of the quantum algebra $U_q[\mathfrak{gl}(3)]$, which is the q-deformed universal enveloping algebra of the Lie algebra $\mathfrak{gl}(3)$ defined below ((22) - (26)).

The idea of deriving of duality relations from the representation matrices of a non-abelian symmetry algebra of a generator of a Markov process goes back to Schütz and Sandow [4] where this strategy was applied to the symmetric partial exclusion process on arbitrary lattices. This is an interacting particle system with a $SU(2)$ symmetry where each lattice site can be occupied by at most a finite number of particles. Next this symmetry approach was extended to prove self-duality of the asymmetric simple exclusion process [5], which is symmetric under the action of quantum algebra $U_q[\mathfrak{gl}(2)]$ and which is an integrable model solvable by Bethe ansatz. The self-duality together with the integrability was used in [6] to

study the time evolution of shock measures and in [7] to study current moments. By mapping the ASEP to a lattice model of interface grwoth the duality function can be interpreted as a lattice Cole-Hopf transformation [5] and is therefore yields information on KPZ interface growth and the moments of the partition function of a directed polymer [8].

The idea of using symmetries of the generator to obtain duality functions was employed again by Giardinà et. al. [9] to study heat conduction in the KMP model with $SU(1,1)$ symmetry and subsequently extended to other interacting particle systems, including particle systems without conservation of particle number [10, 11, 12, 13]. Recently the $U_q[\mathfrak{gl}(2)]$ symmetry was extended to the non-integrable asymmetric generalization of the $SU(2)$-symmetric partial exclusion process [14]. Duality relations for new integrable models that can be solved by Bethe ansatz and related methods were studied very recently in [15, 16].

Here we prove self-duality for the two-component ASEP mentioned above whose symmetry algebra $U_q[\mathfrak{gl}(3)]$ is larger than $SU(2)$, $SU(1,1)$ or their q-deformations. We shall consider only finite systems, the construction and characterization of the properties of the process on $\mathbb{Z}$ is out of the scope of this work. The main novel feature is the presence of more than one conserved species of particles. This leads to interesting non-local properties of the duality functions and, through the integrability of the model, to the possibility of applications in the infinite volume limit employing exact computations along the lines of [17, 18].

The paper is structured as follows. In Sec. 2 we define the process and mention two results obtained in recent work [3] that will be used here. In Sec. 3 we state the main results of the present work. Sec. 4 is included for self-containedness. We describe some tools from linear algebra [19, 20] used in the proofs, which are convenient, but not widely known in the probabilistic treatment of interacting particle systems. In Sec. 5 we present the proofs of our results.

2   The two-component ASEP

We define the process, introduce notation, and mention some results used in the proofs.

2.1   State space and configurations

It is convenient to introduce ternary local state variables $\eta(k) \in \mathbb{S}$ where $\mathbb{S} = \{0, 1, 2\}$. We say that 0 represents occupation of a site a particle of type A, 1 represents a vacant site and 2 represents occupation by a particle of type B. Thus a configuration is denoted by $\boldsymbol{\eta} = \{\eta(-L+1), \ldots, \eta(L)\} \in \mathbb{S}^{2L}$. We call this characterization of a configuration the occupation variable representation. We shall repeatedly consider configurations with a fixed number N particles of type A and M particles of type B. We denote configurations with this property by $\boldsymbol{\eta}_{N,M}$ and the set of all such configurations by $\mathbb{S}^{2L}_{N,M}$.

Equivalently we can specify a configuration $\boldsymbol{\eta}$ uniquely by indicating the particle positions $\mathbf{z}$ on the lattice and write

$$\mathbf{z} = \{\mathbf{x}, \mathbf{y}\} \tag{3}$$

with

$$\mathbf{x} := \{x_i \,:\, \eta(x_i) = 0\}, \quad \mathbf{y} := \{y_i \,:\, \eta(y_i) = 2\} \tag{4}$$

We call this the position representation.

Throughout this work we use the Kronecker-symbol defined by

$$\delta_{\alpha,\beta} = \begin{cases} 1 & \text{if } \alpha = \beta \\ 0 & \text{else} \end{cases} \tag{5}$$

for α, β from any set. We also introduce for $k, l \in \Lambda$

$$\Theta(k, l) := \begin{cases} 1 & k < l \\ 0 & k \geq l \end{cases}. \tag{6}$$

and the indicator function

$$\mathbf{1}_A(\boldsymbol{\eta}) = \begin{cases} 1 & \text{if } \boldsymbol{\eta} \in A \\ 0 & \text{else} \end{cases} \tag{7}$$

for subsets $A \subseteq \mathbb{S}^{2L}$. Some other functions of the configurations will play a role in our treatment:

Definition 2.1 *For $1 \leq k < L$ we define the local permutation*

$$\sigma^{kk+1}(\boldsymbol{\eta}) = \{\eta(-L+1), \ldots \eta(k-1), \eta(k+1), \eta(k), \eta(k+2), \ldots, \eta(L)\} =: \boldsymbol{\eta}^{kk+1}. \tag{8}$$

Definition 2.2 *We define local occupation number variables*

$$a_k(\boldsymbol{\eta}) := \delta_{\eta(k),0}, \quad v_k(\boldsymbol{\eta}) := \delta_{\eta(k),1}, \quad b_k(\boldsymbol{\eta}) := \delta_{\eta(k),2} \tag{9}$$

and the global particle and vacancy numbers

$$N(\boldsymbol{\eta}) = \sum_{k=-L+1}^{L} a_k, \quad M(\boldsymbol{\eta}) = \sum_{k=-L+1}^{L} b_k, \quad V(\boldsymbol{\eta}) = \sum_{k=-L+1}^{L} v_k. \tag{10}$$

The argument of the local occupation number variables will be suppressed throughout this paper, but not the argument of the global particle and vacancy numbers. We note, for $\mathbf{z} = \boldsymbol{\eta}$, the trivial but frequently used identities

$$N(\boldsymbol{\eta}) \equiv N(\mathbf{z}) = |\mathbf{x}|, \quad M(\boldsymbol{\eta}) \equiv M(\mathbf{z}) = |\mathbf{y}|, \tag{11}$$

$$a_k = \sum_{i=1}^{N(\mathbf{z})} \delta_{x_i,k}, \quad b_k = \sum_{i=1}^{M(\mathbf{z})} \delta_{y_i,k}. \tag{12}$$

4

Definition 2.3 *For configurations* $\mathbf{z} = \{\mathbf{x}, \mathbf{y}\}$ *we define the number* $N_k(\mathbf{z})$ *of A-particles to the left of site* $k \in \Lambda$ *and analogously the number* $M_k(\mathbf{z})$ *of B-particles to the left of site* $k \in \Lambda$

$$N_k(\mathbf{z}) := \sum_{l=-L+1}^{k-1} a_l = \sum_{i=1}^{N(\mathbf{z})} \sum_{l=-L+1}^{k-1} \delta_{x_i,l}, \quad M_k(\mathbf{z}) := \sum_{l=-L+1}^{k-1} b_l = \sum_{i=1}^{M(\mathbf{z})} \sum_{l=-L+1}^{k-1} \delta_{y_i,l}, \quad (13)$$

and

$$A_k(\mathbf{z}) := 2N_k(\mathbf{z}) - N(\mathbf{z}), \quad B_k(\mathbf{z}) := 2M_k(\mathbf{z}) - M(\mathbf{z}). \quad (14)$$

Notice that the functions $N(\cdot), N_k(\cdot), a_k(\cdot), A_k(\cdot)$ depend only on the x-coordinates (positions of the A-particles) of a configuration $\mathbf{z}$, while $M(\cdot), M_k(\cdot), b_k(\cdot), B_k(\cdot)$ depend only on the y-coordinates.

2.2 Definition of the two-component ASEP

Recalling the definitions (8) and (12) the two-component ASEP $\boldsymbol{\eta}_t$ described informally in the introduction is defined by the generator

$$\mathcal{L}f(\boldsymbol{\eta}) = \sum_{k=-L+1}^{L-1} w^{kk+1}(\boldsymbol{\eta})[f(\boldsymbol{\eta}^{kk+1}) - f(\boldsymbol{\eta})] \quad (15)$$

with the local hopping rates

$$w^{kk+1}(\boldsymbol{\eta}) = r\left(a_k v_{k+1} + v_k b_{k+1} + a_k b_{k+1}\right) + \ell\left(v_k a_{k+1} + b_k v_{k+1} + b_k a_{k+1}\right) \quad (16)$$

for a transition from a configuration $\boldsymbol{\eta}$ to a configuration $\boldsymbol{\eta}'$ with transition rate

$$w(\boldsymbol{\eta} \to \boldsymbol{\eta}') = \sum_{k=-L+1}^{L-1} w^{kk+1}(\boldsymbol{\eta})\delta_{\boldsymbol{\eta}',\boldsymbol{\eta}^{kk+1}}. \quad (17)$$

It will turn out to be convenient to introduce the asymmetry parameter q and time-scale factor w

$$q = \sqrt{\frac{r}{\ell}}, \quad w = \sqrt{r\ell}. \quad (18)$$

The time scale will play no significant role below.

   The general form of the evolution equation of a Markov chain with state space Ω and transition rates $w(\boldsymbol{\eta} \to \boldsymbol{\eta}')$ for a transition from a configuration $\boldsymbol{\eta} \in \Omega$ to a configuration $\boldsymbol{\eta}' \in \Omega$ is

$$\mathcal{L}f(\boldsymbol{\eta}) = \sum_{\eta' \in \Omega}' w(\eta \to \eta')[f(\eta') - f(\eta)] \quad (19)$$

where the prime at the summation indicates the absence of the term $\eta' = \eta$. We define the transition matrix H of the process by the matrix elements

$$H_{\boldsymbol{\eta}'\boldsymbol{\eta}} = \begin{cases} -w(\boldsymbol{\eta} \to \boldsymbol{\eta}') & \boldsymbol{\eta} \neq \boldsymbol{\eta}' \\ \sum_{\eta'}' w(\boldsymbol{\eta} \to \boldsymbol{\eta}') & \boldsymbol{\eta} = \boldsymbol{\eta}'. \end{cases} \quad (20)$$

The defining equation (19) then becomes

$$\mathcal{L}f(\boldsymbol{\eta}) = -\sum_{\boldsymbol{\eta}'\in\Omega} f(\boldsymbol{\eta}')H_{\boldsymbol{\eta}'\boldsymbol{\eta}} \tag{21}$$

The r.h.s. of (21) represents the multiplication of the matrix H with a vector whose components are $f(\boldsymbol{\eta}')$ in the canonical basis. In slight abuse of language we shall call also H the generator of the process. Below we shall construct H for the two-component exclusion process in a judiciously chosen basis.

2.3   The quantum algebra $U_q[\mathfrak{gl}(n)]$

For the Lie algebra $\mathfrak{gl}(n)$ the quantum algebra $U_q[\mathfrak{gl}(n)]$ is the associative algebra over $\mathbb{C}$ generated by $\mathbf{L}_i^{\pm 1}$, $i = 1, \ldots, n$ and $\mathbf{X}_i^{\pm}$, $i = 1, \ldots, n-1$ with the relations [21, 22]

$$[\,\mathbf{L}_i\,,\,\mathbf{L}_j\,] = 0 \tag{22}$$

$$\mathbf{L}_i\mathbf{X}_j^{\pm} = q^{\pm(\delta_{i,j+1}-\delta_{i,j})/2}\mathbf{X}_j^{\pm}\mathbf{L}_i \tag{23}$$

$$[\,\mathbf{X}_i^+\,,\,\mathbf{X}_j^-\,] = \delta_{ij}\frac{(\mathbf{L}_{i+1}\mathbf{L}_i^{-1})^2 - (\mathbf{L}_{i+1}\mathbf{L}_i^{-1})^{-2}}{q - q^{-1}} \tag{24}$$

and, for $1 \leq i, j \leq n-1$, the quadratic and cubic Serre relations

$$[\,\mathbf{X}_i^{\pm}\,,\,\mathbf{X}_j^{\pm}\,] = 0 \quad |i-j| \neq 1, \tag{25}$$

$$(\mathbf{X}_i^{\pm})^2\mathbf{X}_j^{\pm} - [2]_q\mathbf{X}_i^{\pm}\mathbf{X}_j^{\pm}\mathbf{X}_i^{\pm} + \mathbf{X}_j^{\pm}(\mathbf{X}_i^{\pm})^2 = 0 \quad |i-j| = 1. \tag{26}$$

Here the symmetric q-number is defined by

$$[x]_q := \frac{q^x - q^{-x}}{q - q^{-1}} \tag{27}$$

for q, $q^{-1} \neq 0$ and $x \in \mathbb{C}$. (Notice the replacement $q^2 \to q$ that we made in the definitions of [22].) The notion of symmetry of the generator under the action of the algebra means that there exist representation matrices $Y_i^{\pm}$ and L_j of the algebra that all commute with the transition matrix H of the process, i.e.,

$$[\,H\,,\,Y_i^{\pm}\,] = [\,H\,,\,L_i\,] = 0. \tag{28}$$

For the present case $n = 3$ these representation matrices, given in (108) and (117), were constructed in [3].

3   Results

In order to state the first main result we first define the q-factorial

$$[n]_q! := \begin{cases} 1 & n = 0 \\ \prod_{k=1}^{n}[k]_q & n \geq 1 \end{cases} \tag{29}$$

6

and the q-multinomial coefficients

$$C_K(N) = \frac{[K]_q!}{[N]_q![K-N]_q!}, \quad C_K(N,M) = \frac{[K]_q!}{[N]_q![M]_q![K-N-M]_q!}. \tag{30}$$

Theorem 3.1 *The two-component exclusion process (15) restricted to the subset $\mathbb{S}^{2L}_{N,M}$ of N particles of type A and M particles of type B has the unique invariant measure*

$$\pi^*_{N,M}(\boldsymbol{\eta}) = \frac{\mathbf{1}_{\mathbb{S}^{2L}_{N,M}}(\boldsymbol{\eta})}{Z_{2L}(N,M)}\pi(\boldsymbol{\eta}). \tag{31}$$

with the reversible measure

$$\pi(\boldsymbol{\eta}) = q^{\sum_{k=-L+1}^{L}(2k-1)(a_k-b_k)+\sum_{k=-L+1}^{L-1}\sum_{l=-L+1}^{k}(a_l b_{k+1}-b_l a_{k+1})} \tag{32}$$

and the normalization factor

$$Z_{2L}(N,M) = C_{2L}(N,M). \tag{33}$$

We shall call these invariant measures, characterized in the following theorem, the canonical equilibrium distributions of the process. Particle number conservation yields the following corollary.

Corollary 3.2 *The convex combinations*

$$Q^*_{\nu,\mu}(\boldsymbol{\eta}) = \sum_{N=0}^{2L}\sum_{M=0}^{2L-N}\frac{\mathrm{e}^{\nu N+\mu M}Z_{2L}(N,M)}{Y_{2L}(\nu,\mu)}\pi^*_{N,M}(\boldsymbol{\eta}) = \frac{\mathrm{e}^{\nu N(\boldsymbol{\eta})+\mu M(\boldsymbol{\eta})}}{Y_{2L}(\nu,\mu)}\pi(\boldsymbol{\eta}) \tag{34}$$

with the normalization factor

$$Y_{2L}(\nu,\mu) = \sum_{N=0}^{2L}\sum_{M=0}^{2L-N}\mathrm{e}^{\nu N+\mu M}Z_{2L}(N,M) \tag{35}$$

are invariant measures for the two-component exclusion process (15).

The second equality in (34) follows from the trivial identity $\mathrm{e}^{\nu N+\mu M}\mathbf{1}_{\mathbb{S}^{2L}_{N,M}}(\boldsymbol{\eta}) = \mathrm{e}^{\nu N(\boldsymbol{\eta})+\mu M(\boldsymbol{\eta})}\mathbf{1}_{\mathbb{S}^{2L}_{N,M}}(\boldsymbol{\eta})$. We call these measures the grandcanonical equilibrium distributions. The normalization $Y_{2L}(\nu,\mu)$, is a homogeneous bivariate Rogers-Szegő polynomial [23] and is called the grandcanonical partition function.

The limits $\mu \to -\infty$ or $\nu \to -\infty$ lead to the pure grandcanonical measures

$$Q^{A*}_{\nu}(\boldsymbol{\eta}) = \sum_{N=0}^{2L}\frac{\mathrm{e}^{\nu N}C_{2L}(N)}{X_{2L}(\nu)}\pi^*_{N,0}(\boldsymbol{\eta}) \tag{36}$$

$$Q^{B*}_{\mu}(\boldsymbol{\eta}) = \sum_{M=0}^{2L}\frac{\mathrm{e}^{\mu M}C_{2L}(M)}{X_{2L}(\mu)}\pi^*_{0,M}(\boldsymbol{\eta}) \tag{37}$$

with the Rogers-Szegő polynomial $X_{2L}(\alpha) = \sum_{K=0}^{2L} e^{\alpha K} C_{2L}(K)$. From (31) follows that $\pi_{N,0}^*(\boldsymbol{\eta}) = \mathbf{1}_{\mathbb{S}_{N,0}^{2L}}(\boldsymbol{\eta})\tilde{\pi}_0(\boldsymbol{\eta})/Z_{2L}(N,0)$ with $\tilde{\pi}_0(\boldsymbol{\eta}) = q^{\sum_{k=-L+1}^{L}(2k-1)a_k}$. Since $Z_{2L}(N,0) = C_{2L}(N)$ one finds that $Q_\nu^{A*}(\boldsymbol{\eta})$ is a product measure in $\mathbb{S}_{N,0}^{2L}$ with marginals $Q_\nu^{A*}(k) = (1 + a_k(e^\nu q^{2k-1} - 1))/(e^\nu q^{2k-1} + 1)$, reminiscent of the blocking measure of the single-species ASEP on $\mathbb{Z}$ [2]. Likewise $Q_\mu^{B*}(\boldsymbol{\eta})$ is a product measure in $\mathbb{S}_{0,M}^{2L}$ with marginals $Q_\mu^{B*}(k) = (1 + b_k(e^\mu q^{-2k+1} - 1))/(e^\mu q^{-2k+1} + 1)$. The density profiles $\langle\, a_k \,\rangle_{\nu,0}$ and $\langle\, b_k \,\rangle_{0,\mu}$ in the pure grandcanonical measures follow by straightforward computation. One has shock profiles

$$\langle\, a_k \,\rangle_{\nu,0} \;=\; \frac{e^\nu q^{2k-1}}{1 + e^\nu q^{2k-1}} = \frac{1}{2}\left[1 + \tanh\left(\frac{k - \kappa_A}{\xi}\right)\right] \tag{38}$$

$$\langle\, b_k \,\rangle_{0,\mu} \;=\; \frac{e^\mu q^{-2k+1}}{1 + e^\mu q^{-2k+1}} = \frac{1}{2}\left[1 - \tanh\left(\frac{k - \kappa_B}{\xi}\right)\right] \tag{39}$$

with the shock width $\xi = 1/\ln q$ and shock positions $\kappa_A = (1 - \nu/\ln q)/2$, $\kappa_B = (1 + \nu/\ln q)/2$.

In order to describe the self-duality of the process we define for configurations $\boldsymbol{\eta} \in \mathbb{S}^{2L}$ the functions

$$Q_x^A(\boldsymbol{\eta}) = q^{\sum_{k=-L+1}^{x-1} a_k - \sum_{k=x+1}^{L} a_k} a_x, \quad Q_y^B(\boldsymbol{\eta}) = q^{-\sum_{k=-L+1}^{y-1} b_k + \sum_{k=y+1}^{L} b_k} b_y. \tag{40}$$

From these functions we construct the product

$$Q_{\mathbf{z}}(\boldsymbol{\eta}) := \prod_{i=1}^{N(\mathbf{z})} Q_{x_i}^A(\boldsymbol{\eta}) \prod_{i=1}^{M(\mathbf{z})} Q_{y_i}^B(\boldsymbol{\eta}) \tag{41}$$

indexed by $\mathbf{z} = \{\mathbf{x}, \mathbf{y}\}$, interpreted as a set of coordinates $x_i, y_i \in \Lambda$ and unrelated to $\boldsymbol{\eta}$. With this definition we are in a position to state the second main result of this work.

Theorem 3.3 *Let $\mathbf{z}$ and $\boldsymbol{\eta}$ be two configurations of the two-component exclusion process defined by (15) with asymmetry parameter (18). The process is self-dual with respect to the family of duality functions*

$$D(\mathbf{z}, \boldsymbol{\eta}) = \pi^{-1}(\mathbf{z})Q_{\mathbf{z}}(\boldsymbol{\eta}) \tag{42}$$

where $\pi^{-1}(\mathbf{z})$ is the reversible measure (32).

We remark that the reversible measure (32) can be expressed as

$$\pi(\mathbf{z}) = q^{\sum_{i=1}^{N(\mathbf{z})}[2x_i - 1 - M_{x_i}(\mathbf{z})] - \sum_{i=1}^{M(\mathbf{z})}[(2y_i - 1 - N_{y_i}(\mathbf{z})]} \tag{43}$$

by using (12). Particle number conservation trivially induces independent duality relations for each combination of particle number pairs $(N, M) = (N(\boldsymbol{\eta}), M(\boldsymbol{\eta}))$ and $(N', M') = (N(\mathbf{z}), M(\mathbf{z}))$ with duality functions

$$D_{N,M}^{N',M'}(\mathbf{z}, \boldsymbol{\eta}) := D(\mathbf{z}_{N',M'}, \boldsymbol{\eta}_{N,M})\mathbf{1}_{\mathbb{S}_{N,M}^{2L}}(\boldsymbol{\eta})\mathbf{1}_{\mathbb{S}_{N',M'}^{2L}}(\mathbf{z}). \tag{44}$$

Therefore we refer to a "family of duality functions" rather than just the "duality function". One has $D_{N,M}^{N',M'}(\mathbf{z}, \boldsymbol{\eta}) = 0$ if $N' > N$ or $M' > M$. Using particle number conservation one can construct similar duality functions from $\tilde{Q}_x^A := Q_x^A q^{N(\boldsymbol{\eta})} = q^{2\sum_{k=-L+1}^{x-1} a_k} a_x$ and $\tilde{Q}_y^B := Q_x^B q^{-M(\boldsymbol{\eta})} = q^{-2\sum_{k=-L+1}^{y-1} b_k} b_y$. For the one-component ASEP $\tilde{Q}_x^A$ is the duality function of [5]. A family of duality functions for the ASEP with second-class particles is given by (42) via the replacement $b_k \to v_k$. Duality and reversibility of the process yield the following corollary, see (142):

Corollary 3.4 *For an initial distribution $P_0(\boldsymbol{\eta})$ with an arbitrary number N of particles of type A and M particles of type B we have for the time-dependent expectation*

$$\langle Q_{\mathbf{z}}(t) \rangle_{P_0} = \sum_{z'_{N,M}} \langle Q_{\mathbf{z}'} \rangle_{P_0} F(\mathbf{z}; t|\mathbf{z}'; 0) \qquad (45)$$

where $F(\mathbf{z}; t|\mathbf{z}'; 0)$ is the transition probability of the two-component ASEP with $N = N(\mathbf{z})$ particles of type A and $M = M(\mathbf{z})$ particles of type B.

Explicit exact expressions for $F(\mathbf{z}; t|\mathbf{z}'; 0)$ have been obtained in [18] for the infinite system.

Finally we present some simple properties characterizing the invariant measures. First we remark that by the definition of the process – in which the jumps of the A-particles "do not see" whether the neighbouring site is vacant or occupied by a B-particle – expectations of the form $\langle a_{k_1} \ldots a_{k_n} \rangle_{N,M}$ in the canonical equilibrium measure (31) do not depend on M, i.e., $\langle a_{k_1} \ldots a_{k_n} \rangle_{N,M} = \langle a_{k_1} \ldots a_{k_n} \rangle_{N,0}$ Likewise, $\langle b_{k_1} \ldots b_{k_n} \rangle_{N,M} = \langle b_{k_1} \ldots b_{k_n} \rangle_{0,M}$ does not depend on N.

The second characterization is a sum rule involving the the canonical invariant measures and the duality function.

Theorem 3.5 *Let $\boldsymbol{\eta}_{N,M}$ be a configuration in $\mathbb{S}_{N,M}^{2L}$ with N particles of type A and M particles of type B and let $\mathbf{z}$ be the coordinate representation of a configuration in $\mathbb{S}_{N',M'}^{2L}$ with N' particles of type A and M' particles of type B. Then for all $\mathbf{z} \in \mathbb{S}_{N',M'}^{2L}$ and $\boldsymbol{\eta} \in \mathbb{S}_{N,M}^{2L}$ one has the sum rule*

$$(\pi_{N',M'}^*(\mathbf{z}))^{-1} \sum_{\boldsymbol{\eta}' \in \mathbb{S}_{N,M}^{2L}} \pi_{N,M}^*(\boldsymbol{\eta}') Q_{\mathbf{z}}(\boldsymbol{\eta}') = \sum_{\mathbf{z}' \in \mathbb{S}_{N',M'}^{2L}} Q_{\mathbf{z}'}(\boldsymbol{\eta}) = \lambda_{N,M}^{N',M'} \qquad (46)$$

with a constant $\lambda_{N,M}^{N',M'}$ independent of $\boldsymbol{\eta}$ and $\mathbf{z}$ and canonical stationary distribution given by (31).

We remark that $\lambda_{N,M}^{N',M'} = 0$ if $N' > N$ or $M' > M$.

4 Some tools

4.1 More notation

A generic time-dependent probability measure $\mathrm{Prob}\,[\,\boldsymbol{\eta}_t = \boldsymbol{\eta}\,]$ is denoted by $P(\boldsymbol{\eta}, t)$ or $P(\boldsymbol{\eta}_t)$. For $t = 0$ we use the notation $P_0(\boldsymbol{\eta}) := P(\boldsymbol{\eta}, 0)$. If t is irrelevant we omit the argument t and write $P(\boldsymbol{\eta})$. We also define the transition probability

$$P(\boldsymbol{\eta}', t|\boldsymbol{\eta}, 0) := \mathrm{Prob}\,[\,\boldsymbol{\eta}_t = \boldsymbol{\eta}'|\boldsymbol{\eta}_0 = \boldsymbol{\eta}\,] \tag{47}$$

from a configuration η to a configuration η'.

The expectation of a function $f(\boldsymbol{\eta})$ is denoted by $\langle\, f\, \rangle := \sum_{\boldsymbol{\eta}} f(\boldsymbol{\eta}) P(\boldsymbol{\eta})$. If we specify time and consider an initial distribution $P_0(\boldsymbol{\eta})$ we use for the expectation of a function $f(\boldsymbol{\eta}_t)$ the notation

$$\langle\, f(\boldsymbol{\eta}_t)\, \rangle_{P_0} := \sum_{\boldsymbol{\eta}} P_0(\boldsymbol{\eta}) \sum_{\boldsymbol{\eta}'} f(\boldsymbol{\eta}') P(\boldsymbol{\eta}', t|\boldsymbol{\eta}, 0) \tag{48}$$

or simply $\langle\, f(t)\, \rangle_{P_0}$. For an initial distribution $P_0(\boldsymbol{\eta}') = \delta_{\boldsymbol{\eta}', \boldsymbol{\eta}}$ concentrated on a configuration $\boldsymbol{\eta}$ we write $\langle\, f(\eta_t)\, \rangle_{\boldsymbol{\eta}}$ or $\langle\, f(t)\, \rangle_{\boldsymbol{\eta}}$.

4.2 Matrix form of the generator

It turns out to be convenient to write the generator (15) in the so-called quantum Hamiltonian form [20] which is widely used in the physics literature on stochastic interacting particle systems and which was a given a formal probabilistic description in [19]. However, this approach does not seem to be well-known in the probabilistic literature. For self-containedness and for introduction of our notation we summarize the main ingredients.

4.2.1 Choice of basis, inner product, and tensor product

In order to write the matrix H explicitly one has to choose an concrete basis, i.e., to each configuration $\boldsymbol{\eta}$ one has to assign a specific canonical basis vector. Following [3] we use ternary ordering, i.e., we assign to each configuration $\boldsymbol{\eta}$ the canonical basis vector

$$\iota(\boldsymbol{\eta}) = 1 + \sum_{j=1}^{2L} \eta(j - L) 3^{j-1}. \tag{49}$$

of the complex vector space $\mathbb{C}^d$ with dimension $d = 3^L$. This basis vector has component 1 at position $\iota(\boldsymbol{\eta})$ and 0 else. We work with a vector space over $\mathbb{C}$ rather than over $\mathbb{R}$ since in computations one encounters eigenvectors of H which may be complex.

We denote the basis vectors, which we consider to be column vectors, by $|\,\boldsymbol{\eta}\,\rangle$. We shall also use the notations $|\,\mathbf{z}\,\rangle$ and $|\,\mathbf{x}, \mathbf{y}\,\rangle$ instead of $|\,\boldsymbol{\eta}\,\rangle$. The basis vectors for configurations with a fixed number N of particles of type A and M particles of

type B are denoted by $|\,\boldsymbol{\eta}_{N,M}\,\rangle$. We define also the dual basis $\langle\,\boldsymbol{\eta}\,| = |\,\boldsymbol{\eta}\,\rangle^{T}$, where the superscript T on vectors or matrices denotes transposition.

The inner product of two arbitrary vectors $\langle\,w\,|$ with components w_i and $\langle\,v\,|$ with components v_i is defined by

$$\langle\,w\,|\,v\,\rangle = \sum_{i=1}^{d} w_i v_i \tag{50}$$

without complex conjugation. In particular, we have the biorthogonality relation

$$\langle\,\boldsymbol{\eta}\,|\,\boldsymbol{\eta}'\,\rangle = \delta_{\boldsymbol{\eta}\boldsymbol{\eta}'} \tag{51}$$

Next we introduce the tensor product $|\,v\,\rangle \otimes \langle\,w\,| \equiv |\,v\,\rangle\langle\,w\,|$. This tensor product is a $d \times d$-matrix with matrix elements $(|\,v\,\rangle\langle\,w\,|)_{i,j} = v_i w_j$. Specifically we have the representation

$$\mathbf{1} = \sum_{\boldsymbol{\eta}} |\,\boldsymbol{\eta}\,\rangle\langle\,\boldsymbol{\eta}\,|. \tag{52}$$

of the d-dimensional unit matrix, expressing completeness of the basis.

4.2.2 Measures and expectation values

A probability measure $P(\boldsymbol{\eta})$ is represented by the probability vector

$$|\,P\,\rangle = \sum_{\boldsymbol{\eta}} P(\boldsymbol{\eta})|\,\boldsymbol{\eta}\,\rangle. \tag{53}$$

From the inner product (51) and from (21) we find

$$\mathcal{L}f(\boldsymbol{\eta}) = -\langle\,f\,|H|\,\boldsymbol{\eta}\,\rangle \tag{54}$$

where the vector $\langle\,f\,| = \sum_{\boldsymbol{\eta}} f(\boldsymbol{\eta})\langle\,\boldsymbol{\eta}\,|$ has components $f(\boldsymbol{\eta})$. The semigroup property of the Markov chain is reflected in the time-evolution equation

$$|\,P_t\,\rangle = \mathrm{e}^{-Ht}|\,P_0\,\rangle \tag{55}$$

of a probability measure $P_0(\boldsymbol{\eta})$.

Normalization implies

$$\langle\,s\,|\,P\,\rangle = 1 \tag{56}$$

where the *summation vector*

$$\langle\,s\,| := \sum_{\boldsymbol{\eta}} \langle\,\boldsymbol{\eta}\,| \tag{57}$$

is the row vector where all components are equal to 1. As a consequence one has

$$\langle\,s\,|H = 0 \tag{58}$$

which means that the summation vector is a left eigenvector of H with eigenvector 0. This property follows from the fact that a diagonal element of $H_{\boldsymbol{\eta}\boldsymbol{\eta}}$ is by construction

the sum of all transition rates that appear with negative sign in the same column $\boldsymbol{\eta}$ of H. The vector corresponding to a stationary distribution is denoted $|\pi^*\rangle$. This is a right eigenvector of H with eigenvalue 0:

$$H|\pi^*\rangle = 0. \tag{59}$$

and normalization $\langle s|\pi^*\rangle = 1$. An unnormalized right eigenvector with eigenvalue 0 is denoted $|\pi\rangle$.

The expectation $\langle f \rangle_P$ of a function $f(\boldsymbol{\eta})$ with respect to a probability distribution $P(\boldsymbol{\eta})$ becomes the inner product

$$\langle f \rangle_P = \langle f|P\rangle = \langle s|\hat{f}|P\rangle \tag{60}$$

where

$$\hat{f} := \sum_{\boldsymbol{\eta}} f(\boldsymbol{\eta})|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}| \tag{61}$$

is a diagonal matrix with diagonal elements $f(\boldsymbol{\eta})$. Notice that

$$f(\boldsymbol{\eta}) = \langle\boldsymbol{\eta}|\hat{f}|\boldsymbol{\eta}\rangle = \langle s|\hat{f}|\boldsymbol{\eta}\rangle. \tag{62}$$

For an initial distribution P_0 we can now use the definitions (51), (53), (57) and the representation (52) of the unit matrix to recover (48) in the matrix form

$$
\begin{aligned}
\langle f(t) \rangle_{P_0} &= \sum_{\boldsymbol{\eta}} P_0(\boldsymbol{\eta}) \sum_{\boldsymbol{\eta}'} f(\boldsymbol{\eta}') P(\boldsymbol{\eta}', t|\boldsymbol{\eta}, 0) \\
&= \sum_{\boldsymbol{\eta}'} \langle s|\hat{f}|\boldsymbol{\eta}'\rangle\langle\boldsymbol{\eta}'|\mathrm{e}^{-Ht}|\boldsymbol{\eta}\rangle \\
&= \langle s|\hat{f}\mathrm{e}^{-Ht}|P_0\rangle
\end{aligned}
\tag{63}
$$

Here

$$P(\boldsymbol{\eta}', t|\boldsymbol{\eta}, 0) = \langle\boldsymbol{\eta}'|\mathrm{e}^{-Ht}|\boldsymbol{\eta}\rangle \tag{64}$$

is the transition probability (47).

For a normalized stationary distribution we also define the diagonal matrix

$$\hat{\pi}^* := \sum_{\boldsymbol{\eta}} \pi^*(\boldsymbol{\eta})|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|. \tag{65}$$

For ergodic processes with finite state space one has $0 < \pi^*(\boldsymbol{\eta}) \le 1$ for all $\boldsymbol{\eta}$. Then all powers $(\hat{\pi}^*)^\alpha$ exist. In terms of this diagonal matrix we can write the generator of the reversed dynamics as

$$H^{rev} = \hat{\pi}^* H^T (\hat{\pi}^*)^{-1}. \tag{66}$$

Reversibility means $H^{rev} = H$. An unnormalized stationary distribution π for which

$$H\hat{\pi} = \hat{\pi} H^T \tag{67}$$

holds with

$$\hat{\pi} = \sum_{\boldsymbol{\eta}} \pi(\boldsymbol{\eta})|\boldsymbol{\eta}\rangle\langle\boldsymbol{\eta}|. \tag{68}$$

is called a reversible measure.

4.2.3 Explicit form of the generator

In order to write the generator H explicitly we define the following matrices:

$$a^+ := \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b^+ := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad c^+ := \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad (69)$$

$$a^- := \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad b^- := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad c^- := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (70)$$

the diagonal projectors

$$\hat{a} := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{v} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \hat{b} := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (71)$$

and the three-dimensional unit matrix

$$\mathbb{1} = \hat{a} + \hat{v} + \hat{b}. \quad (72)$$

For matrices M the expression $M^{\otimes j}$ will denote the j-fold tensor product of M with itself if $j > 1$. For $j = 1$ we define $M^{\otimes 1} := M$ and for $j = 0$ we define $M^{\otimes 0} = 1$ with the c-number 1. For arbitrary 3×3-matrices u we define tensor operators

$$u_k := \mathbb{1}^{\otimes(k+L-1)} \otimes u \otimes \mathbb{1}^{\otimes(L-k)} \quad (73)$$

which allow us to write the generator H for the two-component ASEP on the lattice $\{-L+1, \ldots, L\}$ as [3]

$$H = \sum_{k=-L+1}^{L-1} h_{k,k+1} \quad (74)$$

with the hopping matrices

$$\begin{aligned} h_{k,k+1} &:= r\left(\hat{a}_k \hat{v}_{k+1} - a_k^- a_{k+1}^+ + \hat{v}_k \hat{b}_{k+1} - b_k^+ b_{k+1}^- + \hat{a}_k \hat{b}_{k+1} - c_k^- c_{k+1}^+\right) \\ &\quad + \ell\left(\hat{v}_k \hat{a}_{k+1} - a_k^+ a_{k+1}^- + \hat{b}_k \hat{v}_{k+1} - b_k^- b_{k+1}^+ + \hat{b}_k \hat{a}_{k+1} - c_k^+ c_{k+1}^-\right). \end{aligned} \quad (75)$$

With (18) we split $H = H_d + H_o$ into its offdiagonal part

$$H_o = -w \sum_{k=-L+1}^{L-1} \left[q\left(a_k^- a_{k+1}^+ + b_k^+ b_{k+1}^- + c_k^- c_{k+1}^+\right) + q^{-1}\left(a_k^+ a_{k+1}^- + b_k^- b_{k+1}^+ + c_k^+ c_{k+1}^-\right)\right] \quad (76)$$

and its diagonal part

$$H_d = w \sum_{k=-L+1}^{L-1} \left[q\left(\hat{a}_k \hat{v}_{k+1} + \hat{v}_k \hat{b}_{k+1} + \hat{a}_k \hat{b}_{k+1}\right) + q^{-1}\left(\hat{v}_k \hat{a}_{k+1} + \hat{b}_k \hat{v}_{k+1} + \hat{b}_k \hat{a}_{k+1}\right)\right]. \quad (77)$$

For more details of the construction of H in the tensor basis we refer the reader to [3].

4.3   Duality

We recall the concept of duality in matrix form [24, 10], see also [25] for a detailed discussion. In this subsection X and Ω represent arbitrary finite-dimensional state spaces. Consider two processes x_t and ω_t and a function $D : X \times \Omega \mapsto \mathbb{C}$. Notice that the function $D(x, \omega)$ can be understood as a family of functions $f_x : \Omega \mapsto \mathbb{C}$ indexed by x and defined by $f_x(\omega) := D(x, \omega)$, or, alternatively as a family of functions $g_\omega : X \mapsto \mathbb{C}$ indexed by ω and defined by $g_\omega(x) := D(x, \omega)$.

The two processes are said to be dual to each other if

$$\langle\, D(x, \omega_t)\,\rangle_\omega = \langle\, D(x_t, \omega)\,\rangle_x \tag{78}$$

We remark that with the definitions introduced above we have

$$\langle\, D(x, \omega_t)\,\rangle_\omega = \sum_{\omega'} D(x, \omega')P(\omega', t|\omega, 0) = \langle\, f_x(t)\,\rangle_\omega \tag{79}$$

$$\langle\, D(x_t, \omega)\,\rangle_x = \sum_{x'} D(x', \omega)P(x', t|x, 0) = \langle\, g_\omega(t)\,\rangle_x \tag{80}$$

so that duality can be stated as

$$\langle\, f_x(t)\,\rangle_\omega = \langle\, g_\omega(t)\,\rangle_x \tag{81}$$

with $\langle\, f_x(0)\,\rangle_\omega = \langle\, g_\omega(0)\,\rangle_x = D(x, \omega)$.

In order to make contact with the quantum Hamiltonian formalism we define $|\,x\,\rangle$ as a canonical basis vector of $\mathbb{C}^{|X|}$ and $|\,\omega\,\rangle$ as a canonical basis vector of $\mathbb{C}^{|\Omega|}$. Let $\langle\,s\,|$ and $\langle\,\tilde{s}\,|$ be the corresponding summation vectors. Define the matrix

$$D = \sum_x \sum_\omega D(x, \eta)|\,x\,\rangle\langle\,\omega\,| \tag{82}$$

with matrix elements $\langle\,x\,|D|\,\omega\,\rangle = D(x, \omega)$. The processes ω_t and x_t with generators H and G are dual to each other w.r.t. the duality function $D(x, \omega)$ if

$$DH = G^T D. \tag{83}$$

It is easy to prove the equivalence of this definition with the original definition (78). Since the kind of arguments underlying this equivalence are important for the present matrix formulation of duality we present them here in detail:

$$\langle\, D(x, \omega_t)\,\rangle_\omega = \quad \sum_{\omega'} D(x, \omega')P(\omega', t|\omega, 0) \tag{84}$$

$$= \quad \sum_{\omega'} \langle\,x\,|D|\,\omega'\,\rangle\langle\,\omega'\,|\mathrm{e}^{-Ht}|\,\omega\,\rangle \tag{85}$$

$$= \quad \langle\,x\,|D\mathrm{e}^{-Ht}|\,\omega\,\rangle \tag{86}$$

$$= \quad \langle\,x\,|\mathrm{e}^{-G^T t}D|\,\omega\,\rangle \tag{87}$$

$$= \quad \sum_{x'} \langle\,x\,|\mathrm{e}^{-G^T t}|\,x'\,\rangle\langle\,x'\,|D|\,\omega\,\rangle \tag{88}$$

$$= \quad \sum_{x'} D(x', \omega)P(x', t|x, 0) \quad = \langle\, D(x_t, \omega)\,\rangle_x \tag{89}$$

14

In going from (85) to (86) and from (88) to (89) we use the representation of the unit matrix constructed in analogy to (52). In the step from from (86) to (87) we apply the definition (83). Since we have a chain of equalities, it can be read in both directions. Thus the equivalence is established.

In order to express the alternative definition (81) in matrix form we introduce the diagonal matrices

$$\hat{f}_x = \sum_\omega D(x,\omega)|\,\omega\,\rangle\langle\,\omega\,|, \quad \hat{g}_\omega = \sum_x D(x,\omega)|\,x\,\rangle\langle\,x\,|. \tag{90}$$

The duality relation (81) reads

$$\langle\,s\,|\hat{f}_x e^{-Ht}|\,\omega\,\rangle = \langle\,\tilde{s}\,|\hat{g}_\omega e^{-Gt}|\,x\,\rangle. \tag{91}$$

To prove of equivalence of (91) with (83) we note that by construction

$$\langle\,s\,|\hat{f}_x = \langle\,x\,|D, \quad \hat{g}_\omega = |\,\tilde{s}\,\rangle = D|\,\omega\,\rangle. \tag{92}$$

Then it follows that

$$\langle\,s\,|\hat{f}_x e^{-Ht}|\,\omega\,\rangle \quad \langle\,x\,|D e^{-Ht}|\,\omega\,\rangle \tag{93}$$
$$= \langle\,x\,|e^{-G^T t}D|\,\omega\,\rangle \tag{94}$$
$$= \langle\,x\,|e^{-G^T t}\hat{g}_\omega|\,\tilde{s}\,\rangle \quad = \langle\,\tilde{s}\,|\hat{g}_\omega e^{-Gt}|\,x\,\rangle \tag{95}$$

which establishes the equivalence.

We end this discussion with a reformulation of Theorem 2.6 of [10].

Theorem 4.1 *Let H be the matrix representation of the generator of an ergodic Markov process η_t with countable state space and H^{rev} be the matrix form of the generator of the reversed process ξ_t. Assume that there exists an intertwiner S such that*

$$SH = H^{rev}S. \tag{96}$$

Then H is self-dual with duality function $D(\xi,\eta) = D_{\xi,\eta}$ given by the matrix elements of the duality matrix

$$D = \hat{\pi}^{-1}S. \tag{97}$$

with the diagonal stationary distribution matrix (68).

The proof that $SH = H^{rev}S$ implies self-duality with duality matrix $D = (\hat{\pi}^*)^{-1}S$ is elementary and follows from the chain of equalities

$$DH = \hat{\pi}^{-1}SH = \hat{\pi}^{-1}H^{rev}S = \hat{\pi}^{-1}H^{rev}\hat{\pi}D = H^T D. \tag{98}$$

The first and the third equality are the definition (97), the second equality is the hypothesis (96) of the theorem, and the fourth equality is the reversibility relation (67).

Remark 4.2 *It follows that if H is reversible then the hypothesis (96) reads $SH = HS$, i.e. S is a symmetry of H. Unlike [10] we do not require S to be invertible.*

4.4 Representation matrices for $U_q[\mathfrak{gl}(3)]$

4.4.1 Relation between $U_q[\mathfrak{gl}(n)]$ and $U_q[\mathfrak{sl}(n)]$

It is convenient to introduce generators $\mathbf{H}_i$ and $\tilde{\mathbf{H}}_i$ through

$$q^{-\tilde{\mathbf{H}}_i/2} = \mathbf{L}_i, \quad \mathbf{H}_i = \tilde{\mathbf{H}}_i - \tilde{\mathbf{H}}_{i+1}. \tag{99}$$

Then the quantum algebra $U_q[\mathfrak{sl}(n)]$ is the subalgebra generated by $q^{\pm\mathbf{H}_i/2}$, and $\mathbf{X}_i^\pm$, $i = 1, \ldots, n-1$ with relations (25), (26) and

$$q^{\mathbf{H}_i/2}q^{-\mathbf{H}_i/2} = q^{-\mathbf{H}_i/2}q^{\mathbf{H}_i/2} = I \tag{100}$$

$$q^{\mathbf{H}_i/2}q^{\mathbf{H}_j/2} = q^{\mathbf{H}_j/2}q^{\mathbf{H}_i/2} \tag{101}$$

$$q^{\mathbf{H}_i}\mathbf{X}_j^\pm q^{-\mathbf{H}_i} = q^{\pm A_{ij}}\mathbf{X}_j^\pm \tag{102}$$

$$[\mathbf{X}_i^+, \mathbf{X}_j^-] = \delta_{ij}[\mathbf{H}_i]_q. \tag{103}$$

with the unit I and the Cartan matrix A of simple Lie algebras of type A_n

$$A_{ij} := \begin{cases} 2 & i = j \\ -1 & j = i \pm 1 \\ 0 & \text{else.} \end{cases} \tag{104}$$

The fact that $U_q[\mathfrak{sl}(n)]$ is a subalgebra of $U_q[\mathfrak{gl}(n)]$ can be seen by noticing that $\sum_{i=1}^n \tilde{\mathbf{H}}_i$ belongs to the center of $U_q[\mathfrak{gl}_n]$ [21].

4.4.2 Tensor representation for $n = 3$

In order to distinguish the three-dimensional matrices corresponding to the fundamental representation from the abstract generators we use lower case letters. In terms of (69), (70), (71) the fundamental representation of $U_q[\mathfrak{gl}(3)]$ is given by:

$$x_1^\pm = a^\pm, \quad x_2^\pm = b^\mp \tag{105}$$

$$\tilde{h}_1 = \hat{a}, \quad \tilde{h}_2 = \hat{v}, \quad \tilde{h}_3 = \hat{b}, \tag{106}$$

corresponding to

$$h_1 = \hat{a} - \hat{v}, \quad h_2 = \hat{v} - \hat{b}. \tag{107}$$

for the representation of the generators $\mathbf{H}_i$ of $U_q[\mathfrak{sl}(3)]$. It is convenient to work both with h_i and the projectors $\tilde{h}_i$ expressed in term of the projectors (71).

In terms of the fundamental representation a tensor representation of $U_q[\mathfrak{sl}(3)]$, denoted by boldface capital letters, is given by [3]

$$Y_i^\pm = \sum_{k=-L+1}^{L} Y_i^\pm(k) \tag{108}$$

with

$$Y_1^+(k) = q^{\sum_{l=-L+1}^{k-1} \hat{v}_l - \sum_{l=k+1}^{L} \hat{v}_l} a_k^+, \tag{109}$$

16

$$Y_1^-(k) \;=\; q^{-\sum_{l=-L+1}^{k-1}\hat{a}_l+\sum_{l=k+1}^{L}\hat{a}_l}a_k^-, \tag{110}$$

$$Y_2^+(k) \;=\; q^{\sum_{l=-L+1}^{k-1}\hat{b}_l-\sum_{l=k+1}^{L}\hat{b}_l}b_k^- \tag{111}$$

$$Y_2^-(k) \;=\; q^{-\sum_{l=-L+1}^{k-1}\hat{v}_l+\sum_{l=k+1}^{L}\hat{v}_l}b_k^+ \tag{112}$$

and

$$H_i = \sum_{k=-L+1}^{L} H_i(k) \tag{113}$$

with

$$H_i(k) = \mathbb{1}^{\otimes k+L-1} \otimes h_i \otimes \mathbb{1}^{\otimes L-k}. \tag{114}$$

Notice that $H_1(k) = \hat{a}_k - \hat{v}_k$ and $H_2(k) = \hat{v}_k - \hat{b}_k$.

For the full quantum algebra $U_q[\mathfrak{gl}(3)]$ we have the diagonal representation matrices

$$\tilde{H}_1 = \sum_{k=-L+1}^{L} \hat{a}_k =: \hat{N}, \quad \tilde{H}_2 = \sum_{k=-L+1}^{L} \hat{v}_k, \quad \tilde{H}_3 = \sum_{k=-L+1}^{L} \hat{b}_k =: \hat{M}. \tag{115}$$

Here $\hat{N}$ and $\hat{M}$ are the particle number operators satisfying

$$\hat{N}|\,\boldsymbol{\eta}_{N,M}\,\rangle = N|\,\boldsymbol{\eta}_{N,M}\,\rangle, \quad \hat{M}|\,\boldsymbol{\eta}_{N,M}\,\rangle = M|\,\boldsymbol{\eta}_{N,M}\,\rangle. \tag{116}$$

From these matrices one obtains the representation matrices

$$L_i = q^{-\tilde{H}_i/2}. \tag{117}$$

The unit I is represented by the 3^L-dimensional unit matrix $\mathbf{1} := \mathbb{1}^{\otimes 2L}$.

The crucial property of the representation (108) and (117) that was proved in [3] and which is used heavily below are the commutation relations (28) which express the symmetry of the generator H (74) under the action of the quantum algebra $U_q[\mathfrak{gl}(3)]$.

5   Proofs

5.1   Proof of Theorem (3.1)

(i) We first note that uniqueness of $\pi^*_{N,M}$ follows from ergodicity of the process defined on the subset $\mathbb{S}^{2L}_{N,M}$ which is ensured by the fact that the process is a random sequence of permutations $\sigma^{k,k+1}(\boldsymbol{\eta})$.

(ii) In [3] we proved, using the quantum algebra symmetry (28), that the two-component exclusion process defined by (15) has the unnormalized reversible measure π (32). Below we give a direct proof without reference to the quantum algebra symmetry. According to the discussion of Section (4.2) we prove the transformation property (67) with the generator (74). Since $\hat{\pi}$ is diagonal one has has $\hat{\pi}^{-1}H_d\hat{\pi} = H_d$ for the diagonal part (77) of H. It remains to show that $\hat{\pi}^{-1}H_o\hat{\pi} = H_o^T$ for the off-diagonal part (76). To this end we first prove the basic transformation lemma

Lemma 5.1 *For any finite $p \neq 0$ we have*

$$p^{\hat{a}_l} a_x^{\pm} p^{-\hat{a}_l} = p^{\pm \delta_{l,x}} a_x^{\pm}, \quad p^{\hat{b}_l} a_x^{\pm} p^{-\hat{b}_l} = a_x^{\pm}, \tag{118}$$

$$p^{\hat{b}_l} b_x^{\pm} p^{-\hat{b}_l} = p^{\pm \delta_{l,x}} b_x^{\pm}, \quad p^{\hat{a}_l} b_x^{\pm} p^{-\hat{a}_l} = b_x^{\pm} \tag{119}$$

$$p^{\hat{a}_l \hat{b}_m} a_x^{\pm} p^{-\hat{a}_l \hat{b}_m} = p^{\pm \delta_{l,x} \hat{b}_m} a_x^{\pm}, \tag{120}$$

$$p^{\hat{a}_l \hat{b}_m} b_x^{\pm} p^{-\hat{a}_l \hat{b}_m} = p^{\pm \delta_{m,x} \hat{a}_l} b_x^{\pm}. \tag{121}$$

Proof: A projector $\hat{u}$ has the property $\hat{u} = \hat{u}^2$. Thus its exponential can be written $p^{\hat{u}} = 1 + (p-1)\hat{u}$. Since $\hat{a}_l \hat{b}_m$ is a projector one has

$$p^{\hat{a}_l \hat{b}_{k+1}} = 1 + (p-1)\hat{a}_l \hat{b}_{k+1}. \tag{122}$$

The tensor construction implies that $u_k u_l = u_l u_k$ for $k \neq l$ and any u. For $k = l$ we observe that one obtains by direct computation the relations

$$a^+ \hat{a} = b^+ \hat{a} = b^- \hat{a} = c^+ \hat{a} = 0, \quad a^- \hat{a} = a^-, \quad c^- \hat{a} = c^- \tag{123}$$

$$a^- \hat{v} = b^- \hat{v} = c^+ \hat{v} = c^- \hat{v} = 0, \quad a^+ \hat{v} = a^+, \quad b^+ \hat{v} = b^+ \tag{124}$$

$$a^+ \hat{b} = a^- \hat{b} = b^+ \hat{b} = c^- \hat{b} = 0, \quad b^- \hat{b} = b^-, \quad c^+ \hat{b} = c^+ \tag{125}$$

and

$$\hat{a} a^- = \hat{a} b^+ = \hat{a} b^- = \hat{a} c^- = 0, \quad \hat{a} a^+ = a^+, \quad \hat{a} c^+ = c^+ \tag{126}$$

$$\hat{v} a^+ = \hat{a} b^+ = \hat{v} c^+ = \hat{v} c^- = 0, \quad \hat{v} a^- = a^-, \quad \hat{v} b^- = b^- \tag{127}$$

$$\hat{b} a^+ = \hat{b} a^- = \hat{b} b^- = \hat{b} c^+ = 0, \quad \hat{b} b^+ = b^+, \quad \hat{b} c^- = c^-. \tag{128}$$

By multilinearity of the tensor product these relations remain valid on each subspace k. Relations (118) - (121) then follow from (122). $\quad\square$

Now we decompose

$$\hat{\pi} = \hat{A} \hat{B} \hat{U} \tag{129}$$

with

$$\hat{A} = q^{\sum_{k=-L+1}^{L}(2k-1)a_k}, \quad \hat{B} = q^{-\sum_{k=-L+1}^{L}(2k-1)b_k}, \quad \hat{U} = \prod_{k=-L+1}^{L-1} \prod_{l=1}^{k} q^{\hat{a}_l \hat{b}_{k+1} - \hat{b}_l \hat{a}_{k+1}}. \tag{130}$$

Together with $c^{\pm} = a^{\pm} b^{\mp}$ one has from (118) and (119) of Lemma (5.1) for $-L+1 \leq k \leq L$

$$\hat{A} a_k^{\pm} \hat{A}^{-1} = q^{\pm(2k-1)} a_k^{\pm}, \quad \hat{A} b_k^{\pm} \hat{A}^{-1} = b_k^{\pm}, \quad \hat{A} c_k^{\pm} \hat{A}^{-1} = q^{\pm(2k-1)} c_k^{\pm}, \tag{131}$$

$$\hat{B} b_k^{\pm} \hat{B}^{-1} = q^{\mp(2k-1)} b_k^{\pm}, \quad \hat{B} a_k^{\pm} \hat{B}^{-1} = a_k^{\pm}, \quad \hat{B} c_k^{\pm} \hat{B}^{-1} = q^{\pm(2k-1)} c_k^{\pm}. \tag{132}$$

For the transformation U_p one obtains from (120) and (121) of Lemma (5.1) for $-L+1 \leq k \leq L$

$$\hat{U} a_k^{\pm} \hat{U}^{-1} = q^{\mp \sum_{l=-L+1}^{k-1} \hat{b}_l \pm \sum_{l=k+1}^{L} \hat{b}_l} a_k^{\pm} \tag{133}$$

18

$$\hat{U}b_k^\pm\hat{U}^{-1} = q^{\mp\sum_{l=-L+1}^{k-1}\hat{a}_l\pm\sum_{l=k+1}^{L}\hat{a}_l}b_k^\pm \qquad (134)$$

$$\hat{U}c_k^\pm\hat{U}^{-1} = q^{\mp\sum_{l=-L+1}^{k-1}(\hat{b}_l-\hat{a}_l)\pm\sum_{l=k+1}^{L}(\hat{b}_l-\hat{a}_l)}c_k^\pm. \qquad (135)$$

Putting these results together and using the projector property (122) together with (123) - (128) yield for $-L+1 \le k \le L-1$

$$\hat{A}a_k^\pm a_{k+1}^\mp\hat{A}^{-1} = q^{\mp 2}a_k^\pm a_{k+1}^\mp, \ \hat{A}b_k^\pm b_{k+1}^\mp\hat{A}^{-1} = b_k^\pm b_{k+1}^\mp, \ \hat{A}c_k^\pm c_{k+1}^\mp\hat{A}^{-1} = q^{\mp 2}c_k^\pm c_{k+1}^\mp, \qquad (136)$$

$$\hat{B}b_k^\pm b_{k+1}^\mp\hat{B}^{-1} = q^{\pm 2}b_k^\pm b_{k+1}^\mp, \ \hat{B}a_k^\pm a_{k+1}^\mp\hat{B}^{-1} = a_k^\pm a_{k+1}^\mp, \ \hat{B}c_k^\pm c_{k+1}^\mp\hat{B}^{-1} = q^{\mp 2}c_k^\pm c_{k+1}^\mp \qquad (137)$$

and

$$\hat{U}a_k^\pm a_{k+1}^\mp\hat{U}^{-1} = a_k^\pm a_{k+1}^\mp, \ \hat{U}b_k^\pm b_{k+1}^\mp\hat{U}^{-1} = b_k^\pm b_{k+1}^\mp, \ \hat{U}c_k^\pm c_{k+1}^\mp\hat{U}^{-1} = q^{\pm 2}c_k^\pm c_{k+1}^\mp. \qquad (138)$$

Since $(a^\pm)^T = a^\mp$ (and similarly for $b^\pm$ and $c^\pm$) applying the decomposition (129) to the individual terms in (74) yields $\hat{\pi}^{-1}H_o\hat{\pi} = H_o^T$ and therefore reversibility of π.

(iii): We complete the proof of Theorem (3.1) by proving the normalization factor. For a configuration $\mathbf{z} = \{\mathbf{x}, \mathbf{y}\}$ define $\tilde{y}_i == y_i - N_{y_i}(\mathbf{z})$. We have by definition of the partition function

$$Z_{2L}(N, M) = \sum_{\mathbf{z}_{N,M}} \pi(\mathbf{z}_{N,M}) = \sum_{\mathbf{z}_{N,M}} q^{\sum_{i=1}^{N}(2x_i-1)-\sum_{i=1}^{M}(2\tilde{y}_i+N-1)}. \qquad (139)$$

Consider the points $\vec{r}$ in the Weyl alcove $W_K^{2L} = \{\vec{r} : -L < x_1 < ... < x_K \le L\}$. We also define the punctuated Weyl alcove $W_K^{2L}(\vec{r}) = W_K^{2L} \setminus \vec{r}$ for $\vec{r} \in W_K^{2L}$. This allows us to write $\sum_{\mathbf{z}_{N,M}} = \sum_{\vec{x}\in W_N^{2L}} \sum_{\vec{y}\in W_M^{2L}(\vec{x})}$.

Next observe that by construction $\sum_{\vec{y}\in W_M^{2L}(\vec{x})} f(\tilde{y}_i) = \sum_{\vec{y}\in W_M^{2L-N}} f(y_i)$. Therefore

$$Z_{2L}(N, M) = \sum_{\vec{x}\in W_N^{2L}} \sum_{\vec{y}\in W_M^{2L-N}} q^{\sum_{i=1}^{N}(2x_i-1)-\sum_{i=1}^{M}(2y_i+N-1)} \qquad (140)$$

which implies that $Z_{2L}(N, M) = Z_{2L}(N, 0)Z_{2L-N}(0, M)$. A classical result from the theory of integer partitions [26] yields for the single-species partition functions $Z_{2L}(N, 0) = C_{2L}(N)$, $Z_{2L}(0, M)C_{2L}(M)$ with the q-binomial coefficient $C_K(N)$. Observing that $C_{2L}(N)C_{2L-N}(M) = C_{2L}(N, M)$ concludes the proof. $\square$

5.2 Proof of Theorem (3.3)

5.2.1 Reformulation of the problem

Step 1: We first apply the general considerations of Sec. (4.3) to the present case of the two-component ASEP. It is convenient to use the occupation variable presentation $\boldsymbol{\eta}_t$ for one process and the coordinate representation $\mathbf{z}_t$ for the dual. The duality function, given by $\langle \mathbf{z} | D | \boldsymbol{\eta} \rangle$ in terms of the duality matrix D, is therefore

denoted by $D(\mathbf{z}, \boldsymbol{\eta})$. If self-duality is valid for some duality function $D(\mathbf{z}, \boldsymbol{\eta})$ then according to (90) we can define a diagonal matrix $\hat{D}_{\mathbf{z}}$ such that

$$D(\mathbf{z}, \boldsymbol{\eta}) = \langle\, s\, |\hat{D}_{\mathbf{z}}|\, \boldsymbol{\eta}\, \rangle \tag{141}$$

with the summation vector $\langle\, s\, |$. Then self-duality yields

$$\langle\, s\, |\hat{D}_{\mathbf{z}}\mathrm{e}^{-Ht}|\, \boldsymbol{\eta}\, \rangle = \sum_{\mathbf{z}'} \langle\, \mathbf{z}'\, |\mathrm{e}^{-Ht}|\, \mathbf{z}\, \rangle \langle\, s\, |\hat{D}_{\mathbf{z}}|\, \boldsymbol{\eta}\, \rangle \tag{142}$$

and, as a consequence from reversibility, Corollary (45).

Step 2: In [3] we have established the symmetry of the generator under the action of $U_q[\mathfrak{gl}(3)]$. Moreover, we have reversibility $H = H^{rev}$ of the two-component ASEP with the reversible measure (32). Then for any matrix S satisfying $[S, H] = 0$ Theorem (4.1) yields a duality function

$$D(\mathbf{z}, \boldsymbol{\eta}) = \langle\, \mathbf{z}\, |(\hat{\pi}^*(\mathbf{z}))^{-1}S|\, \boldsymbol{\eta}\, \rangle = \pi^{-1}(\mathbf{z})\langle\, \mathbf{z}\, |S|\, \boldsymbol{\eta}\, \rangle. \tag{143}$$

which means that we can construct duality functions from the symmetry operators of the model, i.e., from the tensor representation (108), (113).

Step 3: On the other hand, from (141), one has $D(\mathbf{z}, \boldsymbol{\eta}) = \langle\, s\, |\hat{D}_{\mathbf{z}}|\, \boldsymbol{\eta}\, \rangle$ for all $\boldsymbol{\eta} \in \mathbb{S}^{2L}$. Therefore one can express the duality function

$$D(\mathbf{z}, \boldsymbol{\eta}) = \pi^{-1}(\mathbf{z})Q_{\mathbf{z}}(\boldsymbol{\eta}) = \pi^{-1}(\mathbf{z})\langle\, s\, |\hat{Q}_{\mathbf{z}}|\, \boldsymbol{\eta}\, \rangle. \tag{144}$$

of Theorem (3.3) in terms of a diagonal matrix $\hat{Q}_{\mathbf{z}}$ satisfying

$$\langle\, \mathbf{z}\, |S = \langle\, s\, |\hat{Q}_{\mathbf{z}} \tag{145}$$

and $\langle\, s\, |\hat{Q}_{\mathbf{z}}|\, \boldsymbol{\eta}\, \rangle = Q_{\mathbf{z}}(\boldsymbol{\eta})$ given in (41). Therefore the task at hand is to find a symmetry operator S that satisfies (145) with the diagonal matrix $\hat{Q}_{\mathbf{z}}$ with matrix elements given by (41).

Step 4: In order to choose S we observe that $\hat{D}(\emptyset, \eta) = 1$, corresponding to $\hat{D}_{\emptyset} = \hat{Q}_{\emptyset} = \mathbb{1}$. The non-trivial information one gains is that $\langle\, \emptyset\, |S = \langle\, s\, |$ which means that the symmetry operator S generates the summation vector from the vacuum vector $\langle\, \emptyset\, |$. From the explicit representation obtained in [3] we find as a candidate

$$S = \sum_{n=0}^{2L} \sum_{m=0}^{2L-n} \frac{(Y_1^-)^n}{[n]_q!} \frac{(Y_2^+)^m}{[m]_q!}. \tag{146}$$

Since $\pi(\mathbf{z})$ is known the remaining task is to construct $D(\mathbf{z}, \boldsymbol{\eta})$ as stated in the theorem by proving (145).

5.2.2 Technical lemmas

We prove the following lemmas:

Lemma 5.2 *Consider coordinate sets* $\mathbf{x}' = \mathbf{x} \cup \mathbf{r}$ *and* $\mathbf{y}' = \mathbf{y} \cup \mathbf{s}$. *For* $k \notin \mathbf{r}$ *and* $l \notin \mathbf{s}$ *one has*

$$N_k(\{\mathbf{x} \cup \mathbf{r}, \cdot\}) \;=\; N_k(\{\mathbf{x}, \cdot\}) + N_k(\{\mathbf{r}, \cdot\}) \tag{147}$$

$$\;=\; N_k(\{\mathbf{x}, \cdot\}) + N(\{\mathbf{r}, \cdot\}) - \sum_{i=1}^{N(\{\mathbf{r}, \cdot\})} \Theta(k, r_i) \tag{148}$$

$$M_l(\{\cdot, \mathbf{y} \cup \mathbf{s}\}) \;=\; M_l(\{\cdot, \mathbf{y}\}) + M_l(\{\cdot, \mathbf{s}\}) \tag{149}$$

$$\;=\; M_l(\{\cdot, \mathbf{y}\}) + M(\{\cdot, \mathbf{s}\}) - \sum_{i=1}^{M(\{\cdot, \mathbf{s}\})} \Theta(l, s_i). \tag{150}$$

Proof: The function $\Theta(r, x)$ defined in (6) satisfies

$$\Theta(r, x) = 1 - \Theta(x, r) - \delta_{r,x} \tag{151}$$

$$\sum_{k=-L+1}^{x-1} \delta_{r,k} = \Theta(r, x), \qquad \sum_{k=x+1}^{L} \delta_{r,k} = \Theta(x, r) \tag{152}$$

From (10), (12) and (13) we have

$$\sum_{i=1}^{N(\mathbf{z})} \sum_{l=r+1}^{L} \delta_{x_i,l} \;=\; N(\mathbf{z}) - N_r(\mathbf{z}) - \sum_{i=1}^{N(\mathbf{z})} \delta_{x_i,r} \tag{153}$$

$$\sum_{i=1}^{M(\mathbf{z})} \sum_{l=r+1}^{L} \delta_{y_i,l} \;=\; M(\mathbf{z}) - M_r(\mathbf{z}) - \sum_{i=1}^{M(\mathbf{z})} \delta_{y_i,r}. \tag{154}$$

Specifically, for $N(\mathbf{z}) = 1$ or $M(\mathbf{z}) = 1$ resp. we obtain from (151) - (154)

$$N_r(\{x, \cdot\}) = \Theta(x, r), \quad M_r(y) = \Theta(y, r), \tag{155}$$

and more generally one finds for $\mathbf{z} = \{\mathbf{x}, \mathbf{y}\}$ from (13) and (152)

$$N_r(\{\mathbf{x}, \cdot\}) = \sum_{i=1}^{N(\mathbf{z})} \Theta(x_i, r), \quad M_r(\{\cdot, \mathbf{y}\}) = \sum_{i=1}^{M(\mathbf{z})} \Theta(y_i, r). \tag{156}$$

With (155) and (156) one then finds

$$N_r(\{\mathbf{x}, \cdot\}) = \sum_{i=1}^{N(\mathbf{z})} N_r(\{x_i, \cdot\}), \quad M_r(\{\cdot, \mathbf{y}\}) = \sum_{i=1}^{M(\mathbf{z})} M_r(\{\cdot, y_i\}). \tag{157}$$

The first equality (147) in the lemma then follows from (157). The second equality (148) arises from (151) and (156), bearing in mind that by assumption $k \notin \mathbf{r}$. The proof of (149) and (150) is analogous. $\qquad\square$

In particular, for $\mathbf{x} = \mathbf{y} = \emptyset$ one obtains from Lemma (5.2) for $k \notin \mathbf{r}$ and $l \notin \mathbf{s}$ the inversion formulas

$$N_k(\{\mathbf{r}, \cdot\}) = N(\{\mathbf{r}, \cdot\}) - \sum_{i=1}^{N(\{\mathbf{r}, \cdot\})} N_{r_i}(\{k, \cdot\}) \tag{158}$$

$$M_l(\{\cdot, \mathbf{s}\}) = M(\{\cdot, \mathbf{s}\}) - \sum_{i=1}^{M(\{\cdot, \mathbf{s}\})} M_{s_i}(\{\cdot, l\}). \tag{159}$$

We also derive the projector lemma.

Lemma 5.3 *The tensor occupation operators $\hat{a}_k$, $\hat{b}_k$ act as projectors*

$$\hat{a}_k | \boldsymbol{\eta} \rangle = a_k | \boldsymbol{\eta} \rangle = \sum_{i=1}^{N(\boldsymbol{\eta})} \delta_{x_i, k} | \boldsymbol{\eta} \rangle \tag{160}$$

$$\hat{b}_k | \boldsymbol{\eta} \rangle = b_k | \boldsymbol{\eta} \rangle = \sum_{i=1}^{M(\boldsymbol{\eta})} \delta_{y_i, k} | \boldsymbol{\eta} \rangle \tag{161}$$

with the occupation variables a_k and b_k (9) (or particle coordinates x_i and y_i respectively) understood as functions of $\boldsymbol{\eta}$ or $\mathbf{z} = \boldsymbol{\eta}$.

Proof: The first equality in each equation is inherited from the definition of the projectors (71) by multilinearity of the tensor product, the second equality follows from (12). $\qquad\square$

Finally we note two combinatorial identities for sums over the permutation group S_n. One has

$$\sum_{\sigma \in S_n} q^{-2 \sum_{j=1}^{n} \sum_{i=1}^{j-1} \sigma(\Theta(r_i, r_j)) + n(n-1)/2} = [n]_q! \, q^{-2 \sum_{j=1}^{n} \sum_{i=1}^{j-1} \Theta(r_j, r_i)}, \tag{162}$$

which can be proved by induction using $[n]_q = \sum_{k=0}^{n-1} q^{2k-n+1}$, and

$$\sum_{r_1=-L+1}^{L} \cdots \sum_{r_n=-L+1}^{L} f(r_1, \ldots, r_n) = \sum_{\vec{r}_n} \sum_{\sigma \in S_n} f(\sigma(r_1, \ldots, r_n)) \tag{163}$$

for functions that vanish whenever $r_i = r_j$. Here the sum over $\vec{r}_n$ denotes the summation over the Weyl alcove W_n^{2L}.

5.2.3   Main steps

After these preparations we go on to prove for $\mathbf{z} = \{\mathbf{x}, \mathbf{y}\}$ the property

$$\langle \mathbf{x}, \mathbf{y} | S = \langle s | \left(\prod_{i=1}^{N(\{\mathbf{x}, \mathbf{y}\})} \hat{Q}_{x_i}^A \prod_{i=1}^{M(\{\mathbf{x}, \mathbf{y}\})} \hat{Q}_{y_i}^B \right). \tag{164}$$

The matrices on the r.h.s. are the operator form of the functions $Q'_{A,B}$ defined in (40). According to (145) proving (164) proves the theorem.

From the representation (110), from (160) and from the definition (14) one finds

$$\langle \mathbf{x}, \mathbf{y} | Y_1^-(r) = q^{-A_r(\{\mathbf{x},\mathbf{y}\})} \langle \mathbf{x} \cup r, \mathbf{y} |. \tag{165}$$

By iteration

$$\langle \mathbf{x}, \mathbf{y} | Y_1^-(r_1) \ldots Y_1^-(r_n) = q^{-\sum_{j=1}^n A_{r_j}(\{\mathbf{x} \cup \mathbf{r}_{j-1}, \mathbf{y}\})} \langle \mathbf{x} \cup \mathbf{r}_n, \mathbf{y} | \tag{166}$$

with the definition $\mathbf{r}_0 = \emptyset$. Since from (14), (147), (151) one can write

$$A_{r_j}(\{\mathbf{x} \cup \mathbf{r}_{j-1}, \cdot\}) = 2N_{r_j}(\{\mathbf{x}, \cdot\}) + 2\sum_{i=1}^{j-1} \Theta(r_i, r_j) - (N(\mathbf{x}, \cdot) + j - 1) \tag{167}$$

one has

$$\sum_{j=1}^n A_{r_j}(\{\mathbf{x} \cup \mathbf{r}_{j-1}, \cdot\}) = 2\sum_{j=1}^n N_{r_j}(\{\mathbf{x}, \cdot\}) - nN(\{\mathbf{x}, \cdot\}) + 2\sum_{j=1}^n \sum_{i=1}^{j-1} \Theta(r_i, r_j) - \frac{1}{2}n(n-1). \tag{168}$$

Now we observe that the term $\sum_{j=1}^n (2N_{r_j}(\{\mathbf{x}, \cdot\}) - N(\{\mathbf{x}, \cdot\}))$ is invariant under permutations of the coordinates r_j. Using the fact that $(Y_1^-(r))^2 = 0$ and the combinatorial properties (162) and (163) then yields

$$\langle \mathbf{x}, \mathbf{y} | \frac{(Y_1^-)^n}{[n]_q!} = \sum_{\vec{r}_n} q^{-\sum_{j=1}^n (2N_{r_j}(\{\mathbf{x},\mathbf{y}\}) - N(\{\mathbf{x},\mathbf{y}\}))} \langle \mathbf{x} \cup \mathbf{r}_n, \mathbf{y} |. \tag{169}$$

The next step is to invoke Lemma (5.2) to express $N_{r_j}(\{\mathbf{x}, \cdot\})$ in terms of single-particle step functions $N_{x_i}(\{r_j, \cdot\})$ with inverted arguments. This initiates the following chain of equalities for the exponent $E_\mathbf{r}(\{\mathbf{x}, \cdot\}) := -\sum_{j=1}^n [2N_{r_j}(\{\mathbf{x}, \cdot\}) - N(\{\mathbf{x}, \cdot\})]$ of q:

$$
\begin{aligned}
E_\mathbf{r}(\{\mathbf{x}, \cdot\}) &= -\sum_{j=1}^n \left(N(\{\mathbf{x}, \cdot\}) - 2\sum_{i=1}^{N(\{\mathbf{x},\cdot\})} N_{x_i}(\{r_j, \cdot\}) \right) \\
&= \sum_{j=1}^n \sum_{i=1}^{N(\{\mathbf{x},\cdot\})} (2N_{x_i}(\{r_j, \cdot\}) - 1) \tag{170} \\
&= \sum_{i=1}^{N(\{\mathbf{x},\cdot\})} A_{x_i}(\{\mathbf{r}, \cdot\}).
\end{aligned}
$$

Since trivially

$$\langle \mathbf{x} \cup \mathbf{r}_n, \mathbf{y} | = \langle \mathbf{x} \cup \mathbf{r}_n, \mathbf{y} | a_{x_1} \ldots a_{x_N} \tag{171}$$

23

one arrives at

$$\langle\,\mathbf{x},\mathbf{y}\,|\frac{(Y_1^-)}{[n]_q!} \;=\; \sum_{\vec{r}_n}\left(\prod_{i=1}^{N(\{\mathbf{x},\mathbf{y}\})} q^{A_{x_i}(\{\mathbf{r},\mathbf{y}\})}a_{x_i}\right)\langle\,\mathbf{x}\cup\mathbf{r}_n,\mathbf{y}\,|$$

$$=\; \sum_{\vec{r}_{\tilde{n}}}\left(\prod_{i=1}^{N(\{\mathbf{x},\mathbf{y}\})} q^{A_{x_i}(\{\mathbf{r},\mathbf{y}\})}a_{x_i}\right)\langle\,\mathbf{r}_{\tilde{n}},\mathbf{y}\,|. \tag{172}$$

where the summation in the second equality has been changed to the extended Weyl alcove with $\tilde{n}=N(\{\mathbf{x},\mathbf{y}\})+n$. This is possible since the product of indicators $a_{x_1}\ldots a_{x_N}$ cancels all terms not belonging to the original Weyl alcove n.

Next one uses the definitions (14), (40) and the projector property (160) to express $A_{x_i}(\{\mathbf{r},\mathbf{y}\})$ in terms of projectors. The result is

$$\langle\,\mathbf{x},\mathbf{y}\,|\frac{(Y_1^-)^n}{[n]_q!} \;=\; \sum_{\vec{r}_{\tilde{n}}}\langle\,\mathbf{r}_{\tilde{n}},\mathbf{y}\,|\left(\prod_{i=1}^{N(\{\mathbf{x},\mathbf{y}\})} q^{\sum_{j=1}^{x_i-1}\hat{a}_j-\sum_{j=x_i+1}^{L}\hat{a}_j}\hat{a}_{x_i}\right)$$

$$=\; \sum_{\vec{r}_{\tilde{n}}}\langle\,\mathbf{r}_{\tilde{n}},\mathbf{y}\,|\prod_{i=1}^{N(\{\mathbf{x},\mathbf{y}\})}\hat{Q}_{x_i}^A. \tag{173}$$

Using the commutator relation $[Y_1^-,Y_2^+]=0$ (24) and going through similar steps yields

$$\langle\,\mathbf{x},\mathbf{y}\,|\frac{(Y_1^-)^n}{[n]_q!}\frac{(Y_2^+)^m}{[m]_q!}=\sum_{\vec{r}_{\tilde{n}}}\sum_{\vec{s}_{\tilde{m}}}\langle\,\mathbf{r}_{\tilde{n}},\mathbf{s}_{\tilde{m}}\,|\left(\prod_{i=1}^{N(\{\mathbf{x},\mathbf{y}\})}\hat{Q}_{x_i}^A\prod_{i=1}^{M(\{\mathbf{x},\mathbf{y}\})}\hat{Q}_{y_i}^B\right) \tag{174}$$

with $\tilde{m}=M(\{\mathbf{x},\mathbf{y}\})+m$ and

$$\hat{Q}_y^B=q^{-\sum_{k=-L+1}^{y-1}\hat{b}_k+\sum_{k=y+1}^{L}\hat{b}_k}\hat{b}_y. \tag{175}$$

Observe now that the summation on the r.h.s involves only the vector $\langle\,\mathbf{r}_{\tilde{n}},\mathbf{s}_{\tilde{m}}\,|$. Since the summation is over the Weyl alcove one has

$$\sum_{\vec{r}_{\tilde{n}}}\sum_{\vec{s}_{\tilde{m}}}\langle\,\mathbf{r}_{\tilde{n}},\mathbf{s}_{\tilde{m}}\,|=\langle\,s_{\tilde{n},\tilde{m}}\,|. \tag{176}$$

Therefore

$$\langle\,\mathbf{x},\mathbf{y}\,|\frac{(Y_1^-)^n}{[n]_q!}\frac{(Y_2^+)^m}{[m]_q!}=\langle\,s_{\tilde{n},\tilde{m}}\,|\left(\prod_{i=1}^{N(\{\mathbf{x},\mathbf{y}\})}\hat{Q}_{x_i}^A\prod_{i=1}^{M(\{\mathbf{x},\mathbf{y}\})}\hat{Q}_{y_i}^B\right). \tag{177}$$

Notice that on the r.h.s. the only dependence on the n and m is in the summation vector $\langle\,s_{\tilde{n},\tilde{m}}\,|$ for the sector with $\tilde{n}=N(\{\mathbf{x},\mathbf{y}\})+n$ particles of type A and $\tilde{m}=M(\{\mathbf{x},\mathbf{y}\})+m$ particles of type B.

The final step is to take the double sum (146). Terms such that $n + m > 2L - N(\{\mathbf{x}, \mathbf{y}\}) - M(\{\mathbf{x}, \mathbf{y}\})$ are zero since $\langle s_{N,2L-N} | Y_1^- = \langle s_{N,2L-N} | Y_2^+ = 0$, corresponding to the exclusion principle that forbids creating configurations with more than $2L$ particles on Λ. On the other hand,

$$\langle s_{n,m} | \left(\prod_{i=1}^{N(\{\mathbf{x},\mathbf{y}\})} \hat{Q}_{x_i}^A \prod_{i=1}^{M(\{\mathbf{x},\mathbf{y}\})} \hat{Q}_{y_i}^B \right) = 0 \text{ if } n < N(\{\mathbf{x},\mathbf{y}\}) \text{ or } m < M(\{\mathbf{x},\mathbf{y}\}), \quad (178)$$

due to the projectors contained in the operators $\hat{Q}^{A,B}$. This yields (164) with

$$\hat{Q}_{\mathbf{z}} = \hat{Q}_{\mathbf{x}}^A \hat{Q}_{\mathbf{y}}^B \tag{179}$$

and proves Theorem (3.3) by taking the scalar product with $\langle s |$ and $| \boldsymbol{\eta} \rangle$. $\qquad\square$

5.3   Proof of Theorem (3.5)

The first equality follows from duality and ergodicity by taking the limit $t \to \infty$ in the expectation (45) with coordinate sets $\mathbf{z}'$ representing configurations with N' particles of type A and M' particles of type B.

We also have from (45) by taking $P_0 = \pi_{N,M}^*$, i.e., by considering the canonical equilibrium distribution for N particles of type A and M particles of type B,

$$\langle Q_{\mathbf{z}} \rangle_{\pi_{N,M}^*} = \langle \mathbf{z} | \mathrm{e}^{-Ht} | V_{N,M} \rangle. \tag{180}$$

with a vector

$$| V_{N,M} \rangle = \sum_{\mathbf{z}'} \langle Q_{\mathbf{z}'} \rangle_{\pi_{N,M}^*} | \mathbf{z}' \rangle. \tag{181}$$

that has support in the subspace corresponding to configurations with N' particles of type A and M' particles of type B.

Because of stationarity the l.h.s. does not depend on time. This implies by ergodicity that $| V_{N,M} \rangle$ on the r.h.s. must be proportional to the (unique) stationary probability vector for configurations in $\mathbb{S}_{N',M'}^{2L}$. Hence

$$\langle Q_{\mathbf{z}} \rangle_{\pi_{N,M}^*} = \lambda_{N,M}^{N',M'} \pi_{N',M'}^*(\mathbf{z}) \quad \forall \mathbf{z} \in \mathbb{S}_{N',M'}^{2L} \tag{182}$$

where $\lambda_{N,M}^{N',M'}$ is some constant with $N' = N(\mathbf{z}) = N(\mathbf{z}')$ and $M' = M(\mathbf{z}) = M(\mathbf{z}')$. This proves the second equality in the theorem. $\qquad\square$

Acknowledgements

References

[1] Ferrari, P.A., Kipnis, C., Saada, E.: Microscopic Structure of Travelling Waves in the Asymmetric Simple Exclusion Process. Ann. Probab., **19**(1) 226–244 (1991)

[2] Liggett, T.M.: Interacting particle systems. Springer, Berlin, (1985)

[3] V. Belitsky and G.M. Schütz, Quantum algebra symmetry and reversible measures for the ASEP with second-class particles, (2015)

[4] Schütz, G., Sandow, S.: Non-abelian symmetries of stochastic processes: derivation of correlation functions for random vertex models and disordered interacting many-particle systems. Phys. Rev. E **49**, 2726–2744 (1994)

[5] Schütz, G.M.: Duality relations for asymmetric exclusion processes. J. Stat. Phys. **86**(5/6), 1265–1287 (1997)

[6] Belitsky, V., Schütz, G.M.: Diffusion and coalescence of shocks in the partially asymmetric exclusion process. Electron. J. Prob. **7**, Paper No. 11, 1–21 (2002)

[7] Imamura, T., Sasamoto,T.: Current Moments of 1D ASEP by Duality. J. Stat. Phys. **142**(5), 919–930 (2011)

[8] P. Le Doussal and P. Calabrese, The KPZ equation with flat initial condition and the directed polymer with one free end, J. Stat. Mech. (2012) P06001.

[9] Giardinà, C., Kurchan, J., Redig, F.: Duality and exact correlations for a model of heat conduction. J. Math. Phys. **48**(3), 033301 (2007)

[10] Giardinà, C., Kurchan, J., Redig, F., Vafayi, K.: Duality and Hidden Symmetries in Interacting Particle Systems. J. Stat. Phys. **135**, 25–55 (2009)

[11] J. Ohkubo, Duality in interacting particle systems and boson representation, J. Stat. Phys. **139** 454–465 (2010).

[12] Carinci, G., Giardinà, C., Giberti, C., Redig, F.: Duality for Stochastic Models of Transport J. Stat. Phys. **152**(4), 657–697 (2013)

[13] Carinci, G., Giardinà, C., Giberti, C., Redig, F.: Dualities in population genetics: A fresh look with new dualities. Stochastic Processes Appl. **125**(3), 941–969 (2015).

[14] G. Carinci, C. Giardina, F. Redig, and T. Sasamoto. A generalized Asymmetric Exclusion Process with Uq(sl2) stochastic duality. arXiv:1407.3367 2014. http://arxiv.org/abs/1407.3367, Cited 13 Mar 2015.

[15] Borodin, A., Corwin, I., Sasamoto, T.: From Duality to Determinants for Q-TASEP and ASEP. Ann. Probab. **42**, 2314–2382 (2014).

[16] I. Corwin and L. Petrov, Stochastic higher spin vertex models on the line. arXiv:1502.07374 [math.PR] (2015) http://arxiv.org/abs/1502.07374, Cited 13 Mar 2015.

[17] S. Chatterjee, G. M. Schütz Determinant representation for some transition probabilities in the TASEP with second class particles, J. Stat. Phys. **140**, 900–916 (2010).

[18] C. A. Tracy, H. Widom, On the asymmetric simple exclusion process with multiple species, J. Stat. Phys. **150**, 457–470 (2013).

[19] Lloyd P, Sudbury A, and Donnelly P 1996, Quantum operators in classical probability theory: I. "Quantum spin" techniques and the exclusion model of diffusion, *Stoch. Processes Appl.* **61**(2), 205–221.

[20] G. M. Schütz, Exactly solvable models for many-body systems far from equilibrium, *Phase Transitions and Critical Phenomena.* Vol. 19, Eds. C. Domb and J. Lebowitz, Academic Press, London, (2001).

[21] M. Jimbo, A q-Analogue of $U(\mathfrak{gl}(N+1))$, Hecke Algebra, and the Yang-Baxter Equation, Lett. Math. Phys. **11**, 247-252, 1986

[22] Č. Burdík, M. Havlíček, and A. Vančura, Irreducible Highest Weight Representations of Quantum Groups $U(gl(n, \mathbb{C}))$ Comm. Math. Phys. **148**(2) (1992), 417-423.

[23] Gasper, G., Rahman, M.: Basic hypergeometric series, Encyclopedia of Math. and its Appl. 96 (2nd ed.), Cambridge, Cambridge University Press (2004)

[24] A. Sudbury and P. Lloyd. Quantum operators in classical probability theory. II: The concept of duality in interacting particle systems. Ann. Probab. 23(4) (1995), 1816–1830.

[25] S. Jansen and N. Kurt, On the notion(s) of duality for Markov processes. Prob. Surveys 11 (2014), 59–120.

[26] Andrews, G.E.: The Theory of Partitions, Encyclopedia of Math. and its Appl. 2, London, Addison Wesley (1976)